

Localization and Toeplitz Operators on Polyanalytic Fock Spaces

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Abstract

The well known conjecture of Coburn [L.A. Coburn, *On the Berezin-Toeplitz calculus*, *Proc. Amer. Math. Soc.* 129 (2001) 3331–3338.] proved by Lo [M-L. Lo, *The Bargmann Transform and Windowed Fourier Transform*, *Integr. equ. oper. theory*, 27 (2007), 397–412.] and Engliš [M. Engliš, *Toeplitz Operators and Localization Operators*, *Trans. Am. Math. Society* 361 (2009) 1039–1052.] states that any Gabor-Daubechies operator with window ψ and symbol $\mathbf{a}(x, \omega)$ quantized on the phase space by a Berezin-Toeplitz operator with window Ψ and symbol $\sigma(z, \bar{z})$ coincides with a Toeplitz operator with symbol $D\sigma(z, \bar{z})$ for some polynomial differential operator D .

Using the Berezin quantization approach, we will extend the proof for polyanalytic Fock spaces. While the generation is almost mimetic for two-windowed localization operators, the Gabor analysis framework for vector-valued windows will provide a meaningful generalization of this conjecture for *true polyanalytic* Fock spaces and moreover for polyanalytic Fock spaces.

Further extensions of this conjecture to certain classes of Gel'fand-Shilov spaces will also be considered *a-posteriori*.

Keywords: Localization operators, polyanalytic Fock spaces, Toeplitz operators, Gel'fand-Shilov spaces.

2010 MSC: 47B32, 30H20, 81R30, 81S30, 46F20.

1. Introduction

1.1. State of art

Localization operators rooted in the works of Berezin [9, 10], Shubin [38], Córdoba & Fefferman [19], Daubechies [20], Wong [42] and Ameer, Makarov & Hedenmalm [1] are

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¹N. Faustino was supported by FCT (Portugal) under the fellowship SFRH/BPD/63521/2009 and the project PTDC/MAT/114394/2009. The author is also partially supported by FCT and FEDER (Portugal) under the strategical research project PEst-C/MAT/UI0324/2011 through COMPETE: *Programa Operacional Factores de Competitividade* within QREN.

Preprint submitted to arXiv.org

July 26, 2011

a broad class of anti-Wick operators with a wide range of applications in signal analysis (cf. [21, 36, 23, 18]) and quite recently in random matrix theory (cf. [2, 32]).

The very definition of a localization operator in the language of quantum physics (cf. [26, pp. 193-221]) draws an intuitive construction through coherent states: if we identify each point (x, ω) on the phase space \mathbb{R}^2 as a point $z = x + i\omega$ in the complex plane \mathbb{C} , the quantization of a classical observable $\sigma(z, \bar{z})$ (which is a ultradistribution at best) with respect to the family of classical states $\{|z\rangle : z \in \mathbb{C}\}$ on $L^2(\mathbb{C}, d^2z)$ with duals $\{\langle z| : z \in \mathbb{C}\}$ yields as an integral operator in the Bochner sense defined by

$$S_\sigma = \int_{\mathbb{C}} \sigma(z, \bar{z}) |z\rangle \langle z| d^2z,$$

where $d^2z = \frac{dz d\bar{z}}{2i}$ denotes the symplectic 2-form on \mathbb{C} .

Now let $d\mu(z) = e^{-\pi|z|^2} d^2z$ be the Gaussian measure on \mathbb{C} . The corresponding Hilbert space of square integrable functions with inner product $\langle \cdot, \cdot \rangle_{d\mu}$ and norm $\|\cdot\|_{d\mu} = \langle \cdot, \cdot \rangle_{d\mu}^{\frac{1}{2}}$ will be denoted by $L^2(\mathbb{C}, d\mu)$.

Along this paper we will denote by $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_\omega)$ the standard Cauchy-Riemann operator, by $\partial_z = \frac{1}{2}(\partial_x - i\partial_\omega)$ its conjugate and by Δ_z the Laplace operator $4\partial_{\bar{z}}\partial_z = \partial_x^2 + \partial_\omega^2$. Borrowing from group theoretical backdrop terminology encoded in the Weyl representation W_z :

$$W_z \Psi(\zeta, \bar{\zeta}) = e^{\pi \bar{z} \zeta - \frac{\pi}{2} |z|^2} \Psi(\zeta - z, \bar{\zeta} - \bar{z}) \quad (1)$$

one may interpret families of coherent states as orbit spaces of the given group. In concrete, the Weyl representation (1) is unitary and irreducible on $L^2(\mathbb{C}, d\mu)$ and gives a projective realization for the Heisenberg group \mathbb{H} on $\mathbb{C} \times \mathbb{R}$ endowed with the multiplication rule $(z, t) * (\zeta, \tau) = (z + \zeta, t + \tau + \pi \Im(\bar{z}\zeta))$ (cf. [43]).

In this way one may identify $|z\rangle$ and $\langle z|$ as the action of W_z on the classical states $\Psi, \Theta \in L^2(\mathbb{C}, d\mu)$, that is $|z\rangle \leftrightarrow \langle \cdot, W_z \Psi \rangle_{d\mu}$ and $\langle z| \leftrightarrow \langle W_z \Theta, \cdot \rangle_{d\mu}$. Under this identification the resulting operator S_σ corresponds to the following *Berezin-Toeplitz* operator $\mathcal{L}_\sigma^{\Psi, \Theta}$ with windows $\Psi, \Theta \in L^2(\mathbb{C}, d\mu)$ and symbol $\sigma(z, \bar{z})$:

$$\mathcal{L}_\sigma^{\Psi, \Theta} F = \int_{\mathbb{C}} \sigma(z, \bar{z}) \langle F, W_z \Psi \rangle_{d\mu} W_z \Theta d^2z, \quad \forall F \in L^2(\mathbb{C}, d\mu). \quad (2)$$

This (possibly unbounded) operator defines a localization operator inherit to the Heisenberg group \mathbb{H} (cf. [42], Chapter 17). Further equivalent formulations of S_σ such as *wave packets* (cf. [19]), *Gabor-Daubechies* (cf. [21, 18]) and *Gabor-Toeplitz* operators (cf. [23]) can also be obtained in a similar fashion by replacing the Weyl operator (1) by time-frequency shifts on the phase space \mathbb{R}^2 and/or the anti-Wick operators S_σ by a suitable Weyl pseudo-differential operator. For an overview of Weyl pseudo-differential operators we refer to the books [38, Chapter IV] and [25, Chapter 2 and Chapter 3]. For the connection between Weyl pseudo-differential operators and anti-Wick operators we refer to the papers of *Daubechies* [20] and *Coburn* [14].

In case when $\mathcal{L}_\sigma^{\Psi, \Theta}$ acts on a reproducing kernel Hilbert space $H^2(\mathbb{C}, d\mu)$ of $L^2(\mathbb{C}, d\mu)$ such that $Q : L^2(\mathbb{C}, d\mu) \rightarrow H^2(\mathbb{C}, d\mu)$ is a projection operator, it is therefore naturally to ask in which conditions $\mathcal{L}_\sigma^{\Psi, \Theta}$ and the *Toeplitz* operator $F \mapsto \text{Toep}_\sigma F := Q(\sigma F)$ are equivalent. The main purpose of this statement consists in to get an amalgamation

between function-theoretical and group theoretical machinery with the purpose of comprising the structure of the Segal-Bargmann space and alike encoded on the structure of the reproducing kernels with the irreducibility and square integrability underlying the Weyl representation (1).

This milestone treated on several papers of *Berger & Coburn* (cf. [11, 12, 13, 14, 15]) got some remarkable progress on the papers of *Bauer* [7], *Bauer, Coburn & Isralowitz* [8] and *Coburn, Isralowitz & Li* [17]: In [7] the problem of existence of a *Toeplitz* operator Toep_σ as a product of two *Toeplitz* operators initiated on [14, 15] was further extended to several spaces of (possibly unbounded) smooth symbols including the spaces of measurable functions with certain growth at infinity; on the paper [8] the authors used the heat flow framework of *Berger* and *Coburn* [13] to study the compactness of *Berezin-Toeplitz* operators for certain classes of BMO symbols; in [17] the authors fully characterize *Gabor-Daubechies* operators with BMO symbols using the recent results of *Lo* [34] and *Engliš* [22].

1.2. The Coburn conjecture for analytic Fock spaces

Let us restrict ourselves to the case when $H^2(\mathbb{C}, d\mu)$ is the Fock space $\mathcal{F}(\mathbb{C})$ and Q is the projection operator $P : L^2(\mathbb{C}, d\mu) \rightarrow \mathcal{F}(\mathbb{C})$. For $\Psi = \Theta = \mathbf{1}$ and $\sigma \in L^\infty(\mathbb{C})$, a short calculation shows that Toep_σ and $\mathcal{L}_\sigma^{\mathbf{1}, \mathbf{1}}$ coincide. In case when the constant polynomial $\mathbf{1}$ is replaced by $\Phi_1(z) = \sqrt{\pi}z$ or $\Phi_2(z) = \frac{\pi}{2}z^2$, *Coburn's* result (cf. [15]) under the change of variable $z \mapsto \sqrt{\pi}z$ gives

$$\mathcal{L}_\sigma^{\Phi_1, \Phi_1} = \text{Toep}_{\sigma + \frac{1}{2\pi}\Delta_z\sigma}, \quad \mathcal{L}_\sigma^{\Phi_2, \Phi_2} = \text{Toep}_{\sigma + \frac{1}{\pi}\Delta_z\sigma + 2(\frac{1}{4\pi}\Delta_z)^2\sigma}.$$

The above relations fulfil for every symbol $\sigma(z, \bar{z})$ belonging to the algebra of polynomials $\mathbb{C}[z, \bar{z}]$ or to the algebra $B_a(\mathbb{C})$ of Fourier-Stieltjes transforms with compactly supported measures.

Coburn's most general result conjectured in [15] states that for any $\Psi \in \mathbb{C}[z, \bar{z}] \cap \mathcal{F}(\mathbb{C})$ and $\sigma \in \mathbb{C}[z, \bar{z}] \cup B_a(\mathbb{C})$ there exists a unique polynomial differential operator D depending on $\partial_z, \partial_{\bar{z}}$ and Ψ such that

$$\mathcal{L}_\sigma^{\Psi, \Psi} = \text{Toep}_{D\sigma}. \quad (3)$$

This conjecture was proved at a first glance by *Lo* in [34] when Toep_σ acts solely on *analytic* polynomials on \mathbb{C} . Moreover, using a *molifier scheme* based on the construction of 'cut-off' functions, the author extended relation (3) to a wide class of symbols $E(\mathbb{C})$ including $\mathbb{C}[z, \bar{z}]$ and $B_a(\mathbb{C})$ as well, using mainly dominated convergence results. Hereby

$$E(\mathbb{C}) = \left\{ \sigma \in C^\infty(\mathbb{C}) : \forall k \in \mathbb{N}_0 \exists C, \alpha > 0 \text{ s.t. } |D^k\sigma(z, \bar{z})| \leq Ce^{\alpha|z|}, \forall z \in \mathbb{C} \right\}.$$

An alternative proof of (3) obtained recently by *Engliš* [22] that works for the whole Fock space $\mathcal{F}(\mathbb{C})$ is beyond Wick and anti-Wick correspondence (cf. [25, pp.137-142]). In this context D yields as a Wick ordered operator obtained via the replacements $z \mapsto -\frac{1}{\sqrt{\pi}}\partial_z$ and $\bar{z} \mapsto -\frac{1}{\sqrt{\pi}}\partial_{\bar{z}}$ on the polynomial $D(\bar{z}, z) = e^{\frac{\Delta}{4\pi}}|\Psi(\bar{z}, z)|^2$.

Moreover, under weak assumptions it was shown that $\sigma(z, \bar{z})$ belongs to a broader class of symbols including $BC^\infty(\mathbb{C})$ (space of all C^∞ -functions whose derivatives of all orders are bounded) likewise

$$\mathcal{M}_r = \left\{ \sigma \in C^{2r}(\mathbb{C}) : e^{a|\cdot|} \left| (\partial_{\bar{z}})^l (\partial_z)^m \sigma \right| e^{-\frac{\pi}{2}|\cdot|^2} \in L^\infty(\mathbb{C}), \forall a > 0, \forall 0 \leq l + m \leq 2r \right\}.$$

This later function space contains the class of symbols $\mathbb{C}[z, \bar{z}]$, $B_a(\mathbb{C})$ and $E(\mathbb{C})$.

1.3. Sketch of Results

In this paper we will provide the generalization of *Coburn* conjecture given by equation (3) for *true polyanalytic* Fock spaces/*generalized Bargmann* spaces $\mathcal{F}^j(\mathbb{C})$ of order j (cf. [41, 3]), with $0 \leq j \leq n$, and moreover for the polyanalytic Fock space $\mathbf{F}^n(\mathbb{C}) = L^2(\mathbb{C}, d\mu) \cap \ker(\partial_{\bar{z}})^{n+1}$ of order n .

To be more concise, this framework will be centered around the *Berezin-Toeplitz* operators (2) with windows $\Psi, \Theta \in \mathbf{F}^n(\mathbb{C})$ and symbol $\sigma(z, \bar{z})$ and the family of *Toeplitz* operators Toep_σ^j of order j with symbol $\sigma(z, \bar{z})$ defined as being

$$\text{Toep}_\sigma^j F = P^j(\sigma F) \quad \forall F \in \mathbf{F}^n(\mathbb{C}).$$

Hereby $P^j : L^2(\mathbb{C}, d\mu) \rightarrow \mathcal{F}^j(\mathbb{C})$ denotes the orthogonal projection operator.

In addition, we will denote by $\{\Phi_{j,k}\}_{k \in \mathbb{N}_0}$ the corresponding orthonormal basis of $\mathcal{F}^j(\mathbb{C})$, by $K^j(\zeta, z)$ resp. $\mathbf{K}^n(\zeta, z)$ the reproducing kernel of $\mathcal{F}^j(\mathbb{C})$ resp. $\mathbf{F}^n(\mathbb{C})$. We will write $K(\zeta, z)$ instead of $K^0(\zeta, z) = \mathbf{K}^0(\zeta, z)$ when we refer to the reproducing kernel of the Fock space $\mathcal{F}(\mathbb{C})$. The same nomenclature will be used for $\mathcal{F}^0(\mathbb{C}) = \mathbf{F}^0(\mathbb{C})$ when we refer to $\mathcal{F}(\mathbb{C})$ and analogously to any operator acting on $\mathcal{F}(\mathbb{C})$.

Notice that Toep_σ^j maps $\mathbf{F}^n(\mathbb{C})$ onto $\mathcal{F}^j(\mathbb{C})$. Moreover for $\sigma \in L^\infty(\mathbb{C})$ the boundeness property $\|\text{Toep}_\sigma^j\| \leq \|\sigma\|_{L^\infty(\mathbb{C})}$ is then immediate from construction while the following explicit formula for Toep_σ^j :

$$(\text{Toep}_\sigma^j F)(\zeta) = \int_{\mathbb{C}} \sigma(z, \bar{z}) F(z, \bar{z}) K^j(\zeta, z) d\mu(z) \quad (4)$$

follows straightforwardly from [6, Corollary 7].

The theorem formulated below corresponds to the generalization of *Coburn* conjecture for *true polyanalytic* Fock spaces underlying the class of $BC^\infty(\mathbb{C})$ symbols:

Theorem 1.1. *Let $\Psi \in \mathcal{F}^k(\mathbb{C}) \cap \mathbb{C}[z, \bar{z}]$ and $\Theta \in \mathcal{F}^j(\mathbb{C}) \cap \mathbb{C}[z, \bar{z}]$ such that $\deg(\Psi), \deg(\Theta) < \infty$.*

If $e^{\frac{1}{4\pi}\Delta_z}(\Phi_{k,k}(z, \bar{z})\Phi_{j,j}(z, \bar{z}))$ divides $e^{\frac{1}{4\pi}\Delta_z}(\Psi(z, \bar{z})\overline{\Theta(z, \bar{z})})$ then there exists a polynomial $D_{j,k}(\bar{z}, z)$ of degree $\deg(D_{j,k}) = \deg(\Psi) + \deg(\Theta) - 2j - 2k$ such that for each $\sigma \in BC^\infty(\mathbb{C})$ the operator $D_{j,k} := D_{j,k}\left(-\frac{1}{\sqrt{\pi}}\partial_{\bar{z}}, -\frac{1}{\sqrt{\pi}}\partial_z\right)$ satisfies $D_{j,k}\sigma \in L^\infty(\mathbb{C})$ and

$$\mathcal{L}_\sigma^{\Psi, \Theta} = \text{Toep}_{D_{j,k}\sigma}^j.$$

Moreover $D_{j,k}$ is uniquely determined by

$$D_{j,k} = \frac{e^{\frac{1}{4\pi}\Delta_z} \left(\Psi \left(-\frac{1}{\sqrt{\pi}}\partial_{\bar{z}}, -\frac{1}{\sqrt{\pi}}\partial_z \right) \overline{\Theta \left(-\frac{1}{\sqrt{\pi}}\partial_{\bar{z}}, -\frac{1}{\sqrt{\pi}}\partial_z \right)} \right)}{e^{\frac{1}{4\pi}\Delta_z} \left(\Phi_{k,k} \left(-\frac{1}{\sqrt{\pi}}\partial_{\bar{z}}, -\frac{1}{\sqrt{\pi}}\partial_z \right) \Phi_{j,j} \left(-\frac{1}{\sqrt{\pi}}\partial_{\bar{z}}, -\frac{1}{\sqrt{\pi}}\partial_z \right) \right)}.$$

Although the method of proof is similar to that of [22], the proof of Theorem 1.1 includes the two-windowed case that can belong to *true polyanalytic* Fock spaces of

different orders. In addition it highlights more explicitly the interplay between time-frequency analysis and polyanalytic function spaces tactically described on the papers [5, 6] (see Section 2 of this paper) that in turns yields a meaningful generalization of Theorem 1.1 to the polyanalytic Fock space $\mathbf{F}^n(\mathbb{C})$ (see Corollary 3.7).

Most of the framework performed in Section 3 uses a two-windowed extension of the Berezin symbol/Berezin transform for *generalized Bargmann* spaces (cf. [35, 4]). As we will see in Section 4, this kind of symbols allow us to get a constructive proof for the conjecture providing at the same time a natural extension for a wide class of symbols.

Motivated from the modulation spaces framework developed by *Janssen & Van Eindhoven* [33], *Gröchenig & Zimmermann* [29], *Teofanov* [40] and others we will show in Theorem 4.8 that the Gel'fand-Shilov type spaces resp. tempered ultradistributions introduced in [27] arise naturally as the appropriate symbol classes resp. window classes for studying *Berezin-Toeplitz* operators on polyanalytic Fock spaces.

2. Bargmann-Fock representations for Polyanalytic Fock spaces

2.1. The true polyanalytic Fock spaces revisited

We will explain the construction of *true polyanalytic* Fock spaces using the interrelation between the structure of the Heisenberg group and expansions in terms of special functions using the same order of ideas of *Thangavelu's* book (see [39, Section 1.2]). Similar constructions can be found of the papers of *Askour, Intissar & Mouayn* [3], *Vasilevski* [41] and *Haimi & Hedenmalm* [32, Section 2].

Let us now turn again our attention to the Weyl representation W_ζ defined in (1). The *left invariant* vector-fields associated to W_ζ on \mathbb{H} correspond to the generators I, Z and Z^\dagger of the Lie algebra \mathfrak{h} defined *viz*

$$\begin{aligned} I &: \Psi(z, \bar{z}) \mapsto \Psi(z, \bar{z}) \\ Z &: \Psi(z, \bar{z}) \mapsto \frac{1}{\sqrt{\pi}} \partial_z \Psi(z, \bar{z}) \\ Z^\dagger &: \Psi(z, \bar{z}) \mapsto \sqrt{\pi} z \Psi(z, \bar{z}) - \frac{1}{\sqrt{\pi}} \partial_{\bar{z}} \Psi(z, \bar{z}), \end{aligned} \quad (5)$$

while the *right invariant* vector-fields correspond to the generators I, \bar{Z} and \bar{Z}^\dagger of \mathfrak{h} with

$$\begin{aligned} \bar{Z} &: \Psi(z, \bar{z}) \mapsto \frac{1}{\sqrt{\pi}} \partial_{\bar{z}} \Psi(z, \bar{z}) \\ \bar{Z}^\dagger &: \Psi(z, \bar{z}) \mapsto \sqrt{\pi} \bar{z} \Psi(z, \bar{z}) - \frac{1}{\sqrt{\pi}} \partial_z \Psi(z, \bar{z}). \end{aligned} \quad (6)$$

Therefore $e^{\sqrt{\pi}(\bar{\zeta} Z^\dagger - \zeta Z)} \Psi(z, \bar{z}) = W_\zeta \Psi(z, \bar{z})$ and $e^{\sqrt{\pi}(\zeta \bar{Z}^\dagger - \bar{\zeta} \bar{Z})} \overline{\Psi(z, \bar{z})} = \overline{W_\zeta \Psi(z, \bar{z})}$ follows from direct combination of Taylor series expansion around the point $(\zeta, \bar{\zeta})$ with the direct application of *Baker-Campbell-Hausdorff* formula (cf. [43]):

$$e^R e^S = e^{\frac{1}{2}[R, S]} e^{R+S} \quad \text{whenever} \quad [R, [R, S]] = 0 = [S, [R, S]]. \quad (7)$$

The properties below underlying I, Z and Z^\dagger resp. I, \bar{Z} and \bar{Z}^\dagger follows from construction and from direct application of integration by parts:

i) **Weyl-Heisenberg relations:**

$$\begin{aligned} [Z, Z^\dagger] &= I, & [I, Z] &= 0, & [I, Z^\dagger] &= 0 \\ [\bar{Z}, \bar{Z}^\dagger] &= I, & [I, \bar{Z}] &= 0, & [I, \bar{Z}^\dagger] &= 0. \end{aligned} \quad (8)$$

ii) **Vacuum vector property:** $Z \Phi(z) = 0$ whenever Φ is anti-analytic on \mathbb{C} and $\overline{Z} \Phi(z) = 0$ whenever Φ is analytic on \mathbb{C} .

iii) **Adjoint property:**

$$\langle Z \Phi, \Psi \rangle_{d\mu} = \langle \Phi, Z^\dagger \Psi \rangle_{d\mu} \quad \text{and} \quad \langle \overline{Z} \Phi, \Psi \rangle_{d\mu} = \langle \Phi, \overline{Z}^\dagger \Psi \rangle_{d\mu}. \quad (9)$$

Next, for each $0 \leq j \leq n$, we define the family of subspaces of $\mathbf{F}^n(\mathbb{C})$ resp. $L^2(\mathbb{C}, d\mu)$ using the *Fock* formalism (cf. [24]):

$$\mathcal{F}^j(\mathbb{C}) = \left\{ \frac{1}{\sqrt{j!}} \left(\overline{Z}^\dagger \right)^j \Phi(z) : \Phi \in \ker \overline{Z}, \|\Phi\|_{d\mu} = 1 \right\}.$$

These subspaces are described in terms of the right invariant vector-fields (6) that yield as a direct application of quantum field lemma associated to the second quantization approach (cf. [24]). In particular they are eigenspaces of the magnetic Laplacian $\overline{Z}^\dagger \overline{Z} = \overline{z} \partial_{\overline{z}} - \frac{1}{4\pi} \Delta_z$ with eigenvalue j that include complex Hermite polynomials, complex Laguerre polynomials as well as Fourier expansions of it on $L^2(\mathbb{C}, d\mu)$. Moreover, the corresponding direct sum decompositions:

$$\mathbf{F}^n(\mathbb{C}) = \sum_{j=0}^n \bigoplus \mathcal{F}^j(\mathbb{C}) \quad \text{and} \quad L^2(\mathbb{C}, d\mu) = \sum_{j=0}^\infty \bigoplus \mathcal{F}^j(\mathbb{C}) \quad (10)$$

follow from the fact that the family of subspaces $\{\mathcal{F}^j(\mathbb{C})\}_{0 \leq j \leq n}$ are mutually orthogonal and dense in $L^2(\mathbb{C}, d\mu)$.

2.2. The time-frequency approach

Now we will summarize how the time-frequency analysis framework enters into account in the description of (true) polyanalytic Fock spaces. Most of this results can be found on the papers of *Gröchenig & Lyubarskii* [30, 31] and *Abreu* [5, 6]. Most of the *time-frequency* setting that we will use here and elsewhere is based on the book of *Gröchenig* [28].

For each $\psi \in L^2(\mathbb{R})$, let us denote by $T_x \psi(t) = \psi(t - x)$ a translation by $x \in \mathbb{R}$ by $M_\omega \psi(t) = e^{2\pi i \omega t} \psi(t)$ a modulation by $\omega \in \mathbb{R}$ and by $M_\omega T_x \psi(t) = e^{2\pi i \omega t} \psi(t - x)$ a time-frequency shift by $(x, \omega) \in \mathbb{R}^2$. The short-time Fourier transform (shortly, STFT) with window $\psi \in L^2(\mathbb{R})$ corresponds to

$$(V_\psi f)(x, \omega) = \langle f, M_\omega T_x \psi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(t) \overline{\psi(t - x)} e^{-2\pi i t \omega} dt. \quad (11)$$

This transform possess many structural properties underlying the phase space \mathbb{R}^2 . In particular, the following ones will be useful on the sequel:

Covariance property (cf. [28, Lemma 3.1.3]) Whenever V_ψ is defined, for any $(x, \omega), (u, \eta) \in \mathbb{R}^2$, we have

$$V_\psi(T_u M_\eta f)(x, \omega) = e^{-2\pi i u \omega} V_\psi f(x - u, \omega - \eta). \quad (12)$$

In particular $|V_\psi(T_u M_\eta f)(x, \omega)| = |V_\psi f(x - u, \omega - \eta)|$.

Orthogonality relations (cf. [28, Theorem 3.2.1]) Let $f, g, \phi, \psi \in L^2(\mathbb{R})$. Then $V_\psi f, V_\phi g \in L^2(\mathbb{R}^2)$ and

$$\langle V_\psi f, V_\phi g \rangle_{L^2(\mathbb{R}^2)} = \langle f, g \rangle_{L^2(\mathbb{R})} \overline{\langle \psi, \phi \rangle_{L^2(\mathbb{R})}}. \quad (13)$$

Let us restrict ourselves to the STFT underlying a (normalized) Hermite function of order j as window:

$$h_j(t) = 2^{\frac{1}{4}} j!^{-\frac{1}{2}} e^{\pi t^2} \left(\frac{d}{dt} \right)^j \left(e^{-2\pi t^2} \right), \quad \forall t \in \mathbb{R}. \quad (14)$$

The *true poly-Bargmann transform* of order j defined in the way below (cf. [5, 6])

$$(\mathcal{B}^j f)(x + i\omega) = e^{-i\pi x\omega} e^{\pi \frac{x^2 + \omega^2}{2}} (V_{h_j} f)(x, -\omega) \quad \forall (x, \omega) \in \mathbb{R}^2, \quad (15)$$

corresponds to a meaningful generalization of the Bargmann transform

$$(\mathcal{B}f)(z) = 2^{\frac{1}{4}} \int_{\mathbb{R}} f(t) e^{2\pi tz - \pi t^2 - \frac{\pi}{2} z^2} dt, \quad \forall z \in \mathbb{C}.$$

On the other hand, the covariance property (12) underlying the STFT shows that for each $(u, \eta) \in \mathbb{R}^2$ the time-frequency shift $M_\eta T_u$ and the Bargmann shift $\beta_{u+i\eta} = e^{i\pi u\eta} W_{u-i\eta}$ are intertwined by the *true polyanalytic Bargmann transforms* (15):

$$\beta_{u+i\eta}(\mathcal{B}^j f) = \mathcal{B}^j(M_\eta T_u f), \quad \forall f \in L^2(\mathbb{R}), \quad \forall j = 0, \dots, n. \quad (16)$$

The properties below correspond to a generalization of the results obtained in [28] (see Proposition 3.4.1) and follow straightforwardly by few calculations and by a direct application of the orthogonality relations (13) (cf. [30, 31, 5, 6]).

Proposition 2.1. *If f is a function on \mathbb{R} such that for each $t \in \mathbb{R}$ $|f(t)| = O(|t|^N)$ holds for N sufficiently large, then:*

1. $\mathcal{B}^j f$ is given componentwise by

$$(\mathcal{B}^j f)(z) = (\pi^j j!)^{-\frac{1}{2}} \sum_{l=0}^j \binom{j}{l} (-\pi \bar{z})^{j-l} (\partial_z)^l (\mathcal{B}f)(z).$$

2. The function $z \mapsto (\mathcal{B}^j f)(z)$ is polyanalytic of order $j+1$:

$$(\partial_{\bar{z}})^{j+1} (\mathcal{B}^j f)(z) = 0.$$

3. For any $f, g \in L^2(\mathbb{R})$ we then have $\langle f, g \rangle_{L^2(\mathbb{R})} = \langle \mathcal{B}^j f, \mathcal{B}^j g \rangle_{d\mu}$.
Thus $\mathcal{B}^j : L^2(\mathbb{R}) \rightarrow \mathbf{F}^j(\mathbb{C})$ is an isometry.

As a direct consequence of Proposition 2.1, $\mathcal{B}^j [L^2(\mathbb{R})] = \mathcal{F}^j(\mathbb{C})$ and moreover, the collection of polynomials $\{\Phi_k\}_{k \in \mathbb{N}_0}$ and $\{\Phi_{j,k}\}_{k \in \mathbb{N}_0}$ defined as

$$\Phi_k(z) = \left(\frac{\pi^k}{k!} \right)^{\frac{1}{2}} z^k, \quad \Phi_{j,k}(z, \bar{z}) = (\pi^j j!)^{-\frac{1}{2}} \sum_{l=0}^j \binom{j}{l} (-\pi \bar{z})^{j-l} (\partial_z)^l (\Phi_k(z)) \quad (17)$$

provide a natural basis to the spaces $\mathcal{F}(\mathbb{C})$ and $\mathcal{F}^j(\mathbb{C})$, respectively. Moreover they satisfy $\Phi_k(z) = (\mathcal{B}h_k)(z)$, $\Phi_{0,k}(z, \bar{z}) = \Phi_k(z)$, $\Phi_{j,k}(z, \bar{z}) = (\mathcal{B}^j h_k)(z)$ and the following raising/lowering properties:

$$\begin{aligned} \left(\sqrt{\pi}z - \frac{1}{\sqrt{\pi}}\partial_{\bar{z}} \right) \Phi_{j,k}(z, \bar{z}) &= \sqrt{k+1} \Phi_{j,k+1}(z, \bar{z}) \\ \left(\sqrt{\pi}\bar{z} - \frac{1}{\sqrt{\pi}}\partial_z \right) \Phi_{j,k}(z, \bar{z}) &= -\sqrt{j+1} \Phi_{j+1,k}(z, \bar{z}) \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{1}{\sqrt{\pi}}\partial_z \Phi_{j,k}(z, \bar{z}) &= \sqrt{k} \Phi_{j,k-1}(z, \bar{z}) \\ \frac{1}{\sqrt{\pi}}\partial_{\bar{z}} \Phi_{j,k}(z, \bar{z}) &= -\sqrt{j} \Phi_{j-1,k}(z, \bar{z}). \end{aligned} \quad (19)$$

Combining the Taylor series expansion of the operator $e^{-\frac{1}{4\pi}\Delta_z}$ with the lowering properties (19), one can recast $\Phi_{j,k}(z, \bar{z})$ as a series expansion in terms of the basis functions $\{\Phi_k\}_{k \in \mathbb{N}_0}$ of $\mathcal{F}(\mathbb{C})$:

$$\begin{aligned} \Phi_{j,k}(z, \bar{z}) &= (\pi^j j!)^{-\frac{1}{2}} \sum_{l=0}^j \frac{1}{l!(-\pi)^l} (\partial_{\bar{z}})^l (-\pi\bar{z})^j (\partial_z)^l (\Phi_k(z)) \\ &= \sum_{l=0}^{\infty} \frac{1}{l!(-\pi)^l} (\partial_{\bar{z}})^l (\partial_z)^l (\Phi_j(-\bar{z})\Phi_k(z)) \\ &= e^{-\frac{1}{4\pi}\Delta_z} (\Phi_j(-\bar{z})\Phi_k(z)). \end{aligned} \quad (20)$$

Remark 2.2. The true polyanalytic Bargmann transform of order j (15) is only onto when restricted to the subspace $\mathcal{F}^j(\mathbb{C})$. Moreover the inverse for $\mathcal{B}^j : L^2(\mathbb{R}) \rightarrow \mathcal{F}^j(\mathbb{C})$ is given by the adjoint mapping $(\mathcal{B}^j)^\dagger : \mathcal{F}^j(\mathbb{C}) \rightarrow L^2(\mathbb{R})$.

From the border view of representation theory (see [28, Section 9.2] and references given there) this follows from the fact that the square integrable representation $z \mapsto W_z$ of $L^2(\mathbb{C}, d\mu)$ is reducible on $\mathbf{F}^n(\mathbb{C})$ but irreducible on each $\mathcal{F}^j(\mathbb{C})$.

The mutual orthogonality relations (13) underlying the (normalized) Hermite functions (14) together with (10) shows that for each $F, G \in \mathbf{F}^n(\mathbb{C})$ the inner product $\langle F, G \rangle_{d\mu}$ is uniquely determined by the inner product between the vector-valued functions $\vec{f} = (f_0, f_1, \dots, f_n)$ and $\vec{g} = (g_0, g_1, \dots, g_n)$ on the Hilbert module $L^2(\mathbb{R}; \mathbb{C}^{n+1})$ such that $P^j F = \mathcal{B}^j f_j$ and $P^j G = \mathcal{B}^j g_j$, that is

$$\langle F, G \rangle_{d\mu} = \sum_{j=0}^n \langle \mathcal{B}^j f_j, \mathcal{B}^j g_j \rangle_{d\mu} = \left\langle \vec{f}, \vec{g} \right\rangle_{L^2(\mathbb{R}; \mathbb{C}^{n+1})}.$$

Therefore the isometry $\mathbf{B}^n : L^2(\mathbb{R}; \mathbb{C}^{n+1}) \rightarrow \mathbf{F}^n(\mathbb{C})$ defined viz

$$\mathbf{B}^n \vec{f} = \sum_{j=0}^n \mathcal{B}^j f_j. \quad (21)$$

is rather natural and corresponds to a superposition of the true polyanalytic Bargmann transforms (15).

A short calculation shows that the intertwining properties (16) underlying the time-frequency shifts $M_\eta T_u$ and the Bargmann shifts $\beta_{u+i\eta} = e^{i\pi u\eta} W_{u-i\eta}$ for any $0 \leq j \leq n$ can be extended from linearity to $L^2(\mathbb{R}; \mathbb{C}^{n+1})$. This corresponds to

$$\beta_{u+i\eta}(\mathbf{B}^n \vec{f}) = \mathbf{B}^n(M_\eta T_u \vec{f}), \quad \forall \vec{f} \in L^2(\mathbb{R}; \mathbb{C}^{n+1}). \quad (22)$$

Next, we define the *Gabor-Daubechies* localization operator $\mathcal{A}_{\mathbf{a}}^{\psi, \theta}$ with windows $\psi, \theta \in L^2(\mathbb{R})$ and symbol $\mathbf{a}(x, \omega)$ as being

$$\begin{aligned} \mathcal{A}_{\mathbf{a}}^{\psi, \theta} f &= \int \int_{\mathbb{R}^2} \mathbf{a}(x, \omega) \langle f, M_\omega T_x \psi \rangle_{L^2(\mathbb{R})} M_\omega T_x \theta \, dx d\omega \\ &= \int \int_{\mathbb{R}^2} \mathbf{a}(x, \omega) (V_\psi f)(x, \omega) M_\omega T_x \theta \, dx d\omega. \end{aligned} \quad (23)$$

We will end this section by showing the interplay between the *Gabor-Daubechies* operator (23) and the *Berezin-Toeplitz* operator (2) likewise the boundeness properties for (2) as well.

Lemma 2.3 (see Appendix A). *For any $0 \leq j, k \leq n$ the Gabor-Daubechies operator $\mathcal{A}_{\mathbf{a}}^{\psi, \theta}$ with symbol $\mathbf{a}(x, \omega)$ and windows $\psi, \theta \in L^2(\mathbb{R})$ and the Berezin-Toeplitz operator defined in (2) are interrelated by*

$$\mathcal{L}_\sigma^{\mathbf{B}^k \psi, \mathbf{B}^j \theta} = \mathcal{B}^k \mathcal{A}_{\mathbf{a}}^{\psi, \theta} (\mathcal{B}^j)^\dagger, \quad \text{with } \sigma(z, \bar{z}) = \mathbf{a}(\Re(z), \Im(\bar{z})).$$

Moreover for $\vec{\psi} = (\psi_0, \psi_1, \dots, \psi_n)$ and we $\vec{\theta} = (\theta_0, \theta_1, \dots, \theta_n)$ have

$$\mathcal{L}_\sigma^{\mathbf{B}^n \vec{\psi}, \mathbf{B}^j \theta_j} = \sum_{k=0}^n \mathcal{B}^k \mathcal{A}_{\mathbf{a}}^{\psi_k, \theta} (\mathcal{B}^j)^\dagger \quad \text{and} \quad \mathcal{L}_\sigma^{\mathbf{B}^n \psi, \mathbf{B}^n \theta} = \sum_{j,k=0}^n \mathcal{B}^k \mathcal{A}_{\mathbf{a}}^{\psi_k, \theta} (\mathcal{B}^j)^\dagger.$$

Proposition 2.4. *For any $\Psi, \Theta \in \mathbf{F}^n(\mathbb{C})$, and $\sigma \in L^\infty(\mathbb{C})$ the operator $\mathcal{L}_\sigma^{\Psi, \Theta}$ satisfies the boundeness condition:*

$$\|\mathcal{L}_\sigma^{\Psi, \Theta}\| \leq \|\sigma\|_{L^\infty(\mathbb{C})} \|\Psi\|_{d\mu} \|\Theta\|_{d\mu}.$$

Proof: From Lemma 2.3 and Proposition 2.1 it is equivalent to show the following boundeness condition for $\mathcal{A}_{\mathbf{a}}^{\psi, \theta}$:

$$\|\mathcal{A}_{\mathbf{a}}^{\psi, \theta}\| \leq \|\mathbf{a}\|_{L^\infty(\mathbb{R}^2)} \|\psi\|_{L^2(\mathbb{R})} \|\theta\|_{L^2(\mathbb{R})}. \quad (24)$$

By applying Cauchy-Schwartz inequality to the right-hand side of (23) we get

$$\frac{|\langle \mathcal{A}_{\mathbf{a}}^{\psi, \theta} f, g \rangle_{L^2(\mathbb{R})}|}{\|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}} \leq \|\mathbf{a}\|_{L^\infty(\mathbb{R}^2)} \frac{\|V_\psi f\|_{L^2(\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{R})}} \frac{\|V_\theta g\|_{L^2(\mathbb{R}^2)}}{\|g\|_{L^2(\mathbb{R})}},$$

and hence, from the orthogonality property (13) the right-hand side of the above inequality is equal to $\|\mathbf{a}\|_{L^\infty(\mathbb{R}^2)} \|\psi\|_{L^2(\mathbb{R})} \|\theta\|_{L^2(\mathbb{R})}$. and thus (24) follows from definition. \square

3. Main results

3.1. Reproducing kernels and Berezin symbols

Let us now turn our attention to the reproducing kernel property arising in $\mathcal{F}^j(\mathbb{C})$.

For each $\psi \in L^2(\mathbb{R})$ and $(x, \omega), (u, \eta) \in \mathbb{R}^2$ it follows straightforward from the orthogonality property (13) that $K_\psi(x + i\omega, u + i\eta) = \|\psi\|_{L^2(\mathbb{R})}^{-2} V_\psi(M_\eta T_u \psi)(x, \omega)$ is a reproducing kernel for the Hilbert space $V_\psi(L^2(\mathbb{R}))$:

$$(V_\psi f)(u, \eta) = \langle V_\psi f, K_\psi(\cdot, u + i\eta) \rangle_{L^2(\mathbb{R}^2)}. \quad (25)$$

In the case when ψ corresponds to the (normalized) Hermite function h_j (14), the relation (15) together with the intertwining property (16) regarding the time-frequency shifts $M_{-\eta} T_u$ and the Bargmann shifts $\beta_{u-i\eta} = e^{-i\pi u\eta} W_{u+i\eta}$ enables to reformulate the reproducing kernel property (25) for $\mathcal{F}^j(\mathbb{C})$ in terms of the action of W_ζ on $\Phi_{j,j}(z, \bar{z})$. Namely, the relation

$$e^{i\pi u\eta} e^{-\frac{\pi}{2}(u^2 + \eta^2)} (\mathcal{B}^j f)(u + i\eta) = e^{i\pi u\eta} \langle \mathcal{B}^j f, W_{u+i\eta} \Phi_{j,j} \rangle_{d\mu}$$

combined with the direct sum decompositions (10) yields

$$K^j(\zeta, z) = e^{\frac{\pi}{2}|z|^2} W_z \Phi_{j,j}(\zeta, \bar{\zeta}) = e^{\pi \bar{z}\zeta} \Phi_{j,j}(\zeta - z, \bar{\zeta} - \bar{z}). \quad (26)$$

Remark 3.1. Formula (26) provides a meaningful description for the reproducing kernels obtained in [3, Theorem 3.1], [6, Corollary 5] and [32, Proposition 2.2] using solely the group representation theory framework underlying the Heisenberg group \mathbb{H} .

Remark 3.2. From (26) and the direct sum decompositions (10) the reproducing kernel for $\mathbf{F}^n(\mathbb{C})$ is given by

$$\mathbf{K}^n(z, \zeta) = \sum_{j=0}^n K^j(z, \zeta) = e^{\pi \bar{z}\zeta} \sum_{j=0}^n \Phi_{j,j}(\zeta - z, \bar{\zeta} - \bar{z}).$$

Next we turn our attention for the interplay between the Berezin-Toeplitz operators (2) and the Toeplitz operators (4). The relations below hold in the *weak sense* for any $F \in \mathcal{F}^k(\mathbb{C})$ and $G \in \mathcal{F}^j(\mathbb{C})$:

$$\begin{aligned} \langle \text{Toep}_\sigma^j F, G \rangle_{d\mu} &= \int_{\mathbb{C}} \sigma(z, \bar{z}) F(z, \bar{z}) \overline{G(z, \bar{z})} d\mu(z) \\ &= \int_{\mathbb{C}} \sigma(z, \bar{z}) \langle F, K^k(\cdot, z) \rangle_{d\mu} \langle K^j(\cdot, z), G \rangle_{d\mu} d\mu(z) \\ &= \int_{\mathbb{C}} \sigma(z, \bar{z}) \langle F, W_z \Phi_{k,k} \rangle_{d\mu} \langle W_z \Phi_{j,j}, G \rangle_{d\mu} d^2 z \\ &= \langle \mathcal{L}_\sigma^{\Phi_{k,k}, \Phi_{j,j}} F, G \rangle_{d\mu}. \end{aligned}$$

This combined with (10) give the following identities:

$$\begin{aligned} \text{Toep}_\sigma^j F &= \mathcal{L}_\sigma^{\Phi_{k,k}, \Phi_{j,j}} F && \text{for any } F \in \mathcal{F}^k(\mathbb{C}) \\ \text{Toep}_\sigma^j F &= \sum_{k=0}^n \mathcal{L}_\sigma^{\Phi_{k,k}, \Phi_{j,j}} F && \text{for any } F \in \mathbf{F}^n(\mathbb{C}). \end{aligned} \quad (27)$$

Next, let \mathbf{Op} a bounded linear operator on $\mathbf{F}^n(\mathbb{C})$ and set $K_\zeta^j(z) = K^j(\zeta, \zeta)^{-\frac{1}{2}} K^j(z, \zeta)$ for any $0 \leq j \leq n$. We define $\widetilde{\mathbf{Op}(\zeta)}$ as the $n \times n$ matrix whose entries are given by the Berezin symbols $(\widetilde{\mathbf{Op}(\zeta)})_{j,k} = \left\langle \mathbf{Op} K_\zeta^k, K_\zeta^j \right\rangle_{d\mu}$.

From (20) we get $K^j(\zeta, \zeta) = e^{\pi|\zeta|^2}$ for any $0 \leq j \leq n$. This gives

$$K_\zeta^j(z) = W_\zeta \Phi_{j,j}(z, \bar{z}). \quad (28)$$

On the other hand, since $W_z^\dagger = W_{-z}$ is the adjoint of W_z on $L^2(\mathbb{C}, d\mu)$, it is easy to check from the *Baker-Campbell-Haussdorf* formula (7) the above relations for any $0 \leq j \leq n$:

$$W_z^\dagger K_\zeta^j = e^{i\pi\Im(\bar{\zeta}z)} K_{\zeta-z}^j. \quad (29)$$

The following characterizations for the matrix coefficients defined in (3.3) will be important on the sequel:

Lemma 3.3 (see Appendix A). *For any $\Psi \in \mathcal{F}^k(\mathbb{C})$ and $\Theta \in \mathcal{F}^j(\mathbb{C})$ we have*

$$\left(\widetilde{\mathcal{L}_\sigma^{\Psi, \Theta}(\zeta)} \right)_{j,k} = \left[\sigma * \left(\bar{\Psi} \Theta e^{-\pi|\cdot|^2} \right) \right](\zeta, \bar{\zeta}).$$

Moreover $\left(\widetilde{\mathcal{L}_\sigma^{\Phi_{k,k}, \Phi_{j,j}}(\zeta)} \right)_{j,k} = \left(\widetilde{\text{Toep}_\sigma^j(\zeta)} \right)_{j,k}$.

Proposition 3.4. *Let \mathbf{Op} a bounded linear operator on $\mathbf{F}^n(\mathbb{C})$. Then the following statements hold:*

1. $\left| (\widetilde{\mathbf{Op}(\zeta)})_{j,k} \right| \leq \|\mathbf{Op}\|$ for each $\zeta \in \mathbb{C}$.
2. $\langle \mathbf{Op} F, K_\zeta^j \rangle_{d\mu} = e^{-\frac{\pi}{2}|\zeta|^2} (P^j \mathbf{Op} F)(\zeta, \bar{\zeta})$ for any $F \in \mathbf{F}^n(\mathbb{C})$.
3. \mathbf{Op} is uniquely determined by $\sum_{j,k=0}^n (\widetilde{\mathbf{Op}(\zeta)})_{j,k}$.
4. For any $z \in \mathbb{C}$ we have $\left(\left[W_z^\dagger \mathbf{Op} W_z \right] (\zeta) \right)_{j,k} = (\widetilde{\mathbf{Op}(\zeta + z)})_{j,k}$.

Proof: For the proof of statement (1), we start to recall that W_ζ is unitary on $L^2(\mathbb{C}, d\mu)$ while $\mathcal{B}^j : L^2(\mathbb{R}) \rightarrow \mathcal{F}^j(\mathbb{C})$ is a unitary operator.

Therefore the relations (28) and $\Phi_{j,j} = \mathcal{B}^j h_j$ gives $\left\| K_\zeta^j \right\|_{d\mu} = \|W_\zeta \Phi_{j,j}\|_{d\mu} = 1$, and hence, for any $0 \leq j \leq n$ the Cauchy-Schwartz inequality gives

$$\left| (\widetilde{\mathbf{Op}(\zeta)})_{j,k} \right| \leq \|\mathbf{Op}\| \left\| K_\zeta^k \right\|_{d\mu} \left\| K_\zeta^j \right\|_{d\mu} = \|\mathbf{Op}\|.$$

The proof of statement (2) follows straightforwardly from the identity $\langle \mathbf{Op} F, K^j(\cdot, z) \rangle_{d\mu} = (P^j \mathbf{Op} F)(\zeta, \bar{\zeta})$ and from (28).

For the proof of statement (3), recall that from (29)

$$\begin{aligned} \left\langle \mathbf{Op} K_z^k, K_\zeta^j \right\rangle_{d\mu} &= e^{-\frac{\pi}{2}(|\zeta|^2 + |z|^2)} (P^j \mathbf{Op} K^k(\cdot, z))(\zeta, \bar{\zeta}) \\ &= e^{-\frac{\pi}{2}(|\zeta|^2 + |z|^2)} \overline{(P^k \mathbf{Op}^\dagger K^j(\cdot, \zeta))(z, \bar{z})}. \end{aligned}$$

On the other hand notice that $(P^j \mathbf{Op} K^k(\cdot, z))(\zeta, \bar{\zeta}) = \overline{(P^k \mathbf{Op}^\dagger K^j(\cdot, \zeta))(z, \bar{z})}$ is *true polyanalytic* of order j resp. k in the variable ζ resp. \bar{z} .

Now take $u = \frac{1}{2}(\zeta + \bar{z})$, $\eta = \frac{1}{2i}(\zeta - \bar{z})$ and set $G(u, \eta) = (P^j \mathbf{Op} K^k(\cdot, z))(\zeta, \bar{\zeta})$, for some function G . Then G can be expanded as a Taylor series in the variables u and η whenever $u, \eta \in \mathbb{R}$ (cf. [25, Proposition 1.69]). This implies $z = \zeta$ and hence $\langle \mathbf{Op} K_z^k, K_\zeta^j \rangle_{d\mu}$ is uniquely determined by $(\mathbf{Op}(\zeta))_{j,k}$.

Now recall that each function $F(z, \bar{z})$ belonging to $\mathbf{F}^n(\mathbb{C})$ can be rewritten in terms of the reproducing kernel $\mathbf{K}^n(\zeta, z)$ (see Remark 3.1) and moreover, as a Fourier-Hermite series expansion in terms of the basis functions (17) i.e.

$$F(z, \bar{z}) = \langle F, \mathbf{K}^n(\cdot, z) \rangle_{d\mu} = \sum_{j=0}^n \sum_{l=0}^{\infty} \langle F, \Phi_{j,l} \rangle_{d\mu} \Phi_{j,l}(z, \bar{z}).$$

Thus the normalization of each $K^j(\cdot, z)$ provided by (28) shows that \mathbf{Op} is uniquely determined by $\sum_{j,k=0}^n (\mathbf{Op}(\zeta))_{j,k}$, as desired (cf. [25, Corollary 1.70]).

Finally, the proof of statement (4) yields from direct application of the property (29) in terms of $-z$. since $W_z^\dagger = W_{-z}$ is the adjoint of W_z on $L^2(\mathbb{C}, d\mu)$. \square

Remark 3.5. From Lemma 3.3, the coefficients $\left(\widetilde{\mathcal{L}_\sigma^{\Psi, \Theta}(\zeta)} \right)_{j,j}$ of the $n \times n$ matrix $\widetilde{\mathcal{L}_\sigma^{\Psi, \Theta}(\zeta)}$ correspond to the two-windowed generalization of the magnetic Berezin transform attached to the true polyanalytic Fock spaces $\mathcal{F}^j(\mathbb{C})$ (cf. [35, 4]) whereas Proposition 3.4 gives a generalization of [22, Proposition 3] for $\mathbf{F}^n(\mathbb{C})$.

3.2. Proof of Coburn Conjecture for Polyanalytic Fock spaces

For the proof of Theorem 1.1, the following lemma will be required *a-posteriori*:

Lemma 3.6 (see Appendix A). For each $m \in \mathbb{N}$ we have the following intertwining properties on $L^2(\mathbb{C}, d\mu)$:

$$(\pi z - \partial_{\bar{z}})^m = e^{-\frac{1}{4\pi}\Delta z} (\pi z)^m e^{\frac{1}{4\pi}\Delta z}, \quad (\pi \bar{z} - \partial_z)^m = e^{-\frac{1}{4\pi}\Delta \bar{z}} (\pi \bar{z})^m e^{\frac{1}{4\pi}\Delta \bar{z}}$$

Proof: [Proof of Theorem 1.1] From (2), we obtain by (29) the relation

$$\begin{aligned} \mathcal{L}_{\sigma(\cdot+z, \cdot+\bar{z})}^{\Psi, \Theta} &= \int_{\mathbb{C}} \sigma(\zeta + z, \bar{\zeta} + \bar{z}) \langle \cdot, W_\zeta \Psi \rangle_{d\mu} W_\zeta \Theta d^2 \zeta \\ &= \int_{\mathbb{C}} \sigma(z, \bar{z}) \langle \cdot, W_{\zeta-z} \Psi \rangle_{d\mu} W_{\zeta-z} \Theta d^2 \zeta \\ &= W_z^\dagger \mathcal{L}_\sigma^{\Psi, \Theta} W_z. \end{aligned}$$

In particular, from (27) we then have $\text{Toep}_{\sigma(\cdot+z, \cdot+\bar{z})}^j = W_z^\dagger \text{Toep}_\sigma^j W_z$.

From Proposition 2.4 and statements (1),(2) and (4) of Proposition 3.4, for each $\sigma \in L^\infty(\mathbb{C})$ the Berezin symbols $\left(\widetilde{\mathcal{L}_\sigma^{\Psi, \Theta}(\zeta)} \right)_{j,k}$ and $\left(\text{Toep}_\sigma^j(\zeta) \right)_{j,k}$ are bounded above by bounded *true polyanalytic* functions of order j which are invariant under translations.

Then for each $\Psi \in \mathcal{F}^k(\mathbb{C}) \cap \mathbb{C}[z, \bar{z}]$ and $\Theta \in \mathcal{F}^j(\mathbb{C}) \cap \mathbb{C}[z, \bar{z}]$ the distributions $\overline{\Psi(z, \bar{z})}\Theta(z, \bar{z})e^{-\pi|z|^2}$ and moreover $\Phi_{k,k}(z, \bar{z})\Phi_{j,j}(z, \bar{z})e^{-\pi|z|^2}$ satisfying Lemma 3.3 are uniquely determined (cf. [37, Theorem 6.33]).

Now let $D_{j,k}(z, \bar{z})$ be a polynomial of degree $N = \deg(\Psi) + \deg(\Theta) - 2j - 2k$ written in terms of sequence of polynomials $\{\Phi_k\}_{k \in \mathbb{N}_0}$ of $\mathcal{F}(\mathbb{C})$ defined in (17):

$$D_{j,k}(z, \bar{z}) = \sum_{l+m=0}^N d_{l,m} \Phi_l(z) \Phi_m(\bar{z}).$$

Notice that the adjoint properties (9) on $L^2(\mathbb{C}, d\mu)$ combined with Lemma 3.6 gives the sequence of identities

$$\begin{aligned} & \left\langle (-\partial_{\bar{z}})^m (-\partial_z)^l \sigma(\zeta - \cdot, \overline{\zeta - \cdot}), \Phi_{k,k} \Phi_{j,j} \right\rangle_{d\mu} = \\ & = \left\langle \sigma(\zeta - \cdot, \overline{\zeta - \cdot}), (\pi z - \partial_{\bar{z}})^l (\pi \bar{z} - \partial_z)^m (\Phi_{k,k}(z, \bar{z}) \Phi_{j,j}(z, \bar{z})) \right\rangle_{d\mu} \\ & = \left\langle \sigma(\zeta - \cdot, \overline{\zeta - \cdot}), e^{-\frac{1}{4\pi}\Delta_z} (\pi z)^l (\pi \bar{z})^m e^{\frac{1}{4\pi}\Delta_z} (\Phi_{k,k}(z, \bar{z}) \Phi_{j,j}(z, \bar{z})) \right\rangle_{d\mu}. \end{aligned}$$

Using linearity arguments we then have

$$\begin{aligned} & \left\langle D_{j,k} \left(-\frac{1}{\sqrt{\pi}} \partial_{\bar{z}}, -\frac{1}{\sqrt{\pi}} \partial_z \right) \sigma(\zeta - \cdot, \overline{\zeta - \cdot}), \Phi_{k,k}(z, \bar{z}) \Phi_{j,j}(z, \bar{z}) \right\rangle_{d\mu} = \\ & = \left\langle \sigma(\zeta - \cdot, \overline{\zeta - \cdot}), e^{-\frac{1}{4\pi}\Delta_z} D_{j,k}(z, \bar{z}) e^{\frac{1}{4\pi}\Delta_z} (\Phi_{k,k}(z, \bar{z}) \Phi_{j,j}(z, \bar{z})) \right\rangle_{d\mu}. \end{aligned}$$

Therefore from Lemma 3.3 $(\widetilde{\mathcal{L}_\sigma^{\Psi, \Theta}(\zeta)})_{j,k} = (\widetilde{\text{Toep}_{D_{j,k}\sigma}^j(\zeta)})_{j,k}$ if and only if the following identity holds almost everywhere in \mathbb{C} :

$$\Psi(z, \bar{z}) \overline{\Theta(z, \bar{z})} = e^{-\frac{1}{4\pi}\Delta_z} D_{j,k}(z, \bar{z}) e^{\frac{1}{4\pi}\Delta_z} (\Phi_{k,k}(z, \bar{z}) \Phi_{j,j}(z, \bar{z})). \quad (30)$$

Multiplying both sides of the above equation on the left by the operator $e^{-\frac{1}{4\pi}\Delta_z}$, the polynomial $D_{j,k}(\bar{z}, z)$ is uniquely determined by

$$D_{j,k}(\bar{z}, z) = \frac{e^{\frac{1}{4\pi}\Delta_z} (\Psi(\bar{z}, z) \overline{\Theta(\bar{z}, z)})}{e^{\frac{1}{4\pi}\Delta_z} (\Phi_{k,k}(\bar{z}, z) \Phi_{j,j}(\bar{z}, z))}$$

only when $e^{\frac{1}{4\pi}\Delta_z} (\Phi_{k,k}(z, \bar{z}) \Phi_{j,j}(z, \bar{z}))$ divides $e^{\frac{1}{4\pi}\Delta_z} (\Psi(z, \bar{z}) \overline{\Theta(z, \bar{z})})$.

Therefore, under the constraint $\sigma \in BC^\infty(\mathbb{C})$, from statement (3) of Proposition 3.4 the polynomial differential operator $D_{j,k} := D_{j,k} \left(-\frac{1}{\sqrt{\pi}} \partial_{\bar{z}}, -\frac{1}{\sqrt{\pi}} \partial_z \right)$ satisfies $\mathcal{L}_\sigma^{\Psi, \Theta} = \text{Toep}_{D_{j,k}\sigma}^j$ with $D_{j,k}\sigma \in L^\infty(\mathbb{C})$. \square

The extension of Theorem 1.1 to the Fock space $\mathbf{F}^n(\mathbb{C})$ is now straightforward from the direct sum decompositions (10).

Corollary 3.7. Let $\Psi, \Theta \in \mathbf{F}^n(\mathbb{C}) \cap \mathbb{C}[z, \bar{z}]$ and $P^j : L^2(\mathbb{C}, d\mu) \rightarrow \mathcal{F}^j(\mathbb{C})$, $P^k : L^2(\mathbb{C}, d\mu) \rightarrow \mathcal{F}^k(\mathbb{C})$ the corresponding projection operators.

If $e^{\frac{1}{4\pi}\Delta_z} (\Phi_{k,k}(z, \bar{z}) \Phi_{j,j}(z, \bar{z}))$ divides $e^{\frac{1}{4\pi}\Delta_z} \left((P^k \Psi)(z, \bar{z}) \overline{(P^j \Theta)(z, \bar{z})} \right)$ then there exists a unique polynomial differential operator $D_j := D_j \left(-\frac{1}{\sqrt{\pi}} \partial_{\bar{z}}, -\frac{1}{\sqrt{\pi}} \partial_z \right)$ such that

1. $D_j(\bar{z}, z)$ has degree

$$\deg(D_j) = \max_{0 \leq k \leq n} (\deg(P^k \Psi) + \deg(P^j \Theta) - 2j - 2k).$$

2. $D_j \sigma \in L^\infty(\mathbb{C})$.

3. $\mathcal{L}_\sigma^{\Psi, \Theta} = \sum_{j=0}^n \text{Toep}_{D_j \sigma}^j$.

Proof: Let $\Psi, \Theta \in \mathbf{F}^n(\mathbb{C}) \cap \mathbb{C}[z, \bar{z}]$. The finite expansion in terms of the projection operators $P^j : L^2(\mathbb{C}, d\mu) \rightarrow \mathcal{F}^j(\mathbb{C})$ resp. $P^k : L^2(\mathbb{C}, d\mu) \rightarrow \mathcal{F}^k(\mathbb{C})$ yielding from the direct sum decompositions (10) gives

$$\mathcal{L}_\sigma^{\Psi, \Theta} = \sum_{j,k=0}^n \mathcal{L}_\sigma^{P^k \Psi, P^j \Theta}.$$

From hypothesis $e^{\frac{1}{4\pi}\Delta_z} (\Phi_{k,k}(z, \bar{z}) \Phi_{j,j}(z, \bar{z}))$ divides $e^{\frac{1}{4\pi}\Delta_z} \left((P^k \Psi)(z, \bar{z}) \overline{(P^j \Theta)(z, \bar{z})} \right)$, under the hypothesis of Theorem 1.1 there exists a unique polynomial differential operator $D_{j,k}$ with symbol

$$D_{j,k}(\bar{z}, z) = \frac{e^{\frac{1}{4\pi}\Delta_z} \left((P^k \Psi)(\bar{z}, z) \overline{(P^j \Theta)(\bar{z}, z)} \right)}{e^{\frac{1}{4\pi}\Delta_z} (\Phi_{k,k}(\bar{z}, z) \Phi_{j,j}(\bar{z}, z))}$$

such that $D_{j,k} \sigma \in L^\infty(\mathbb{C})$ has degree $\deg(D_{j,k}) = \deg(P^k \Psi) + \deg(P^j \Theta) - 2j - 2k$ and satisfies

$$\mathcal{L}_\sigma^{P^k \Psi, P^j \Theta} = \text{Toep}_{D_{j,k} \sigma}^j.$$

Thus for $D_j := \sum_{k=0}^n D_{j,k}$, the polynomial $D_j(\bar{z}, z) := \sum_{k=0}^n D_{j,k}(\bar{z}, z)$ has degree $\deg(D_j) = \max_{0 \leq k \leq n} \deg(D_{j,k})$ and the later equation is equivalent to

$$\mathcal{L}_\sigma^{\Psi, \Theta} = \sum_{j=0}^n \text{Toep}_{D_j \sigma}^j, \quad \text{with } D_j \sigma \in L^\infty(\mathbb{C}).$$

□

The next corollary which is then immediate from Theorem 1.1 a mimic generalization of a result obtained by *Engliš* (cf. [22, Corollary 4]) to the polyanalytic Fock space $\mathbf{F}^n(\mathbb{C})$ following also from the same order of ideas used on the proof of Corollary 3.7.

Corollary 3.8. Let $\Psi, \Psi^*, \Theta, \Theta^* \in \mathbf{F}^n(\mathbb{C}) \cap \mathbb{C}[z, \bar{z}]$. Then the following statements are equivalent:

(a) There exist a unique sequence of polynomial differential operators $\{D_{j,k}\}_{0 \leq j,k \leq n}$ with Wick symbols $D_{j,k}(\bar{z}, z)$ such that

$$\mathcal{L}_\sigma^{\Psi, \Theta} = \sum_{j,k=0}^n \mathcal{L}_{D_{j,k}\sigma}^{P^k\Psi^*, P^j\Theta^*}. \quad (31)$$

(b) $e^{\frac{1}{4\pi}\Delta z} \left((P^k\Psi^*)(z, \bar{z}) \overline{(P^j\Theta^*)(z, \bar{z})} \right)$ divides $e^{\frac{1}{4\pi}\Delta z} \left((P^k\Psi)(z, \bar{z}) \overline{(P^j\Theta)(z, \bar{z})} \right)$.

Whence, if (a) or (b) fulfils the Wick symbol $D_{j,k}(\bar{z}, z)$ has degree $\deg(D_{j,k}) = \deg(P^j\Theta) + \deg(P^k\Psi) - \deg(P^j\Theta^*) - \deg(P^k\Psi^*)$ and (31) holds for every $\sigma \in BC^\infty(\mathbb{C})$.

4. Extension to a Wide Class of Symbols

4.1. Gel'fand-Shilov type Spaces

According to the proof of Theorem 1.1 and subsequent corollaries in Subsection 3.2, $D_{j,k}$ is a *anti-Wick ordered operator* constructed as bijective mapping of the set of polynomials $\mathbb{C}[z, \bar{z}]$ onto the set of differential operators with polynomial coefficients whereas the condition $\sigma \in BC^\infty(\mathbb{C})$ assures that for each $0 \leq j, k \leq n$ the symbols $D_{j,k}\sigma$ belongs to $L^\infty(\mathbb{C})$.

Motivated by the framework described by Lo (cf. [34]) for the spaces $B_a(\mathbb{C})$ and $E(\mathbb{C})$ and by Engliš (cf. [22]) for the spaces \mathcal{M}_r , we will introduce a new family of function spaces that constitute a rich class of symbols including $E(\mathbb{C})$ and \mathcal{M}_r as well.

For $a > 0$, $1 \leq p \leq \infty$ and $\frac{1}{2} \leq \alpha \leq 1$, we introduce the function spaces $\mathcal{W}_{a,\alpha}^{p,n}$, $\mathcal{G}_n^{\{\alpha\}}$ and $\mathcal{G}_n^{(\alpha)}$ as follows:

i) $\sigma \in \mathcal{W}_{a,\alpha}^{p,n}$ if and only if for every $0 \leq j \leq n$ and $l, m \in \mathbb{N}_0$ we have

$$e^{a|\cdot|^{\frac{1}{\alpha}}} e^{-\frac{\pi}{2}|\cdot|^2} P^j \sigma \in L^p(\mathbb{C}).$$

ii) $\sigma \in \mathcal{G}_n^{\{\alpha\}}$ if and only if there exists $a > 0$ such that $\sigma \in \mathcal{W}_{a,\alpha}^{p,n}$.

iii) $\sigma \in \mathcal{G}_n^{(\alpha)}$ if and only if $\sigma \in \mathcal{W}_{a,\alpha}^{p,n}$ for every $a > 0$.

In case when $\sigma \in \mathcal{W}_{a,\alpha}^{p,n}$, the quantity

$$\|\sigma\|_{\mathcal{W}_{a,\alpha}^{p,n}} := \sum_{j=0}^n \left\| e^{a|\cdot|^{\frac{1}{\alpha}}} e^{-\frac{\pi}{2}|\cdot|^2} P^j \sigma \right\|_{L^p(\mathbb{C})}$$

is a *quasi-norm* for $\mathcal{W}_{a,\alpha}^{p,n}$ while $\mathcal{G}_n^{\{\alpha\}}$ resp. $\mathcal{G}_n^{(\alpha)}$ are the complex analogues of the Gel'fand-Shilov spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R})$ resp. $\Sigma_\alpha^\alpha(\mathbb{R})$ whereas its dual spaces $\mathcal{G}_n^{\{\alpha\}'}$ resp. $\mathcal{G}_n^{(\alpha)'}$ are the complex analogues of the spaces of tempered ultradistributions $\mathcal{S}_\alpha^\alpha(\mathbb{R})'$ resp. $\Sigma_\alpha^\alpha(\mathbb{R})'$ of Beurling resp. Romieu type (cf. [27]).

From the following reformulation of a result of Gröchenig and Zimmermann ([29, Proposition 4.3]) for the classes of Gel'fand-Shilov spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R})$ resp. $\Sigma_\alpha^\alpha(\mathbb{R})$ and tempered ultradistributions $\mathcal{S}_\alpha^\alpha(\mathbb{R})'$ resp. $\Sigma_\alpha^\alpha(\mathbb{R})'$ one can prove that they are indeed isomorphic. In terms of the *true polyanalytic* Bargmann transforms (15), this theorem is stated as follows:

Theorem 4.1. Let $\frac{1}{2} \leq \alpha \leq 1$. Then for each $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R})'$ (resp. for each $f \in \Sigma_\alpha^\alpha(\mathbb{R})'$) and for each $j \in \mathbb{N}_0$ the following two conditions are equivalent:

- i) $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R})$ (resp. $f \in \Sigma_\alpha^\alpha(\mathbb{R})$)
- ii) There exists $b, c > 0$ (resp. for every $b, c > 0$) such that

$$e^{b|x|^{\frac{1}{\alpha}} + c|\omega|^{\frac{1}{\alpha}}} (\mathcal{B}^j f)(x + i\omega) e^{-\frac{\pi}{2}(x^2 + \omega^2)} \in L^\infty(\mathbb{R}^2).$$

Proposition 4.2. We have the following isometric isomorphisms:

$$\begin{aligned} \mathcal{G}^{\{\alpha\}} &\cong \bigotimes_{0 \leq j \leq n} \mathcal{S}_\alpha^\alpha(\mathbb{R}) \quad \text{and} \quad \mathcal{G}^{(\alpha)} \cong \bigotimes_{0 \leq j \leq n} \Sigma_\alpha^\alpha(\mathbb{R}) \\ \mathcal{G}^{\{\alpha\}'} &\cong \bigotimes_{0 \leq j \leq n} \mathcal{S}_\alpha^\alpha(\mathbb{R})' \quad \text{and} \quad \mathcal{G}^{(\alpha)'} \cong \bigotimes_{0 \leq j \leq n} \Sigma_\alpha^\alpha(\mathbb{R})'. \end{aligned}$$

Proof: Using the fact that for any $\sigma \in L^\infty(\mathbb{C})$ such that $(P^j \sigma)(z) = (\mathcal{B}^j f_j)(x + i\omega)$, the quantities $\left\| e^{|\cdot|^{\frac{1}{\alpha}}} e^{-\frac{\pi}{2}|\cdot|^2} (P^j \sigma) \right\|_{L^\infty(\mathbb{C})}$ and $\left\| e^{a|\cdot|^{\frac{1}{\alpha}} + b|\cdot|^{\frac{1}{\alpha}}} e^{-\frac{\pi}{2}((\cdot)^2 + (\cdot)^2)} (\mathcal{B}^j f_j)(\cdot + i\cdot) \right\|_{L^\infty(\mathbb{R}^2)}$ endow equivalent norms, we show that $\sigma \in \mathcal{G}_n^{\{\alpha\}}$ resp. $\sigma \in \mathcal{G}_n^{(\alpha)}$ if and only if the vector $\vec{f} = (f_0, f_1, \dots, f_n)$ belongs to $\bigotimes_{0 \leq j \leq n} \mathcal{S}_\alpha^\alpha(\mathbb{R})$ resp. $\bigotimes_{0 \leq j \leq n} \Sigma_\alpha^\alpha(\mathbb{R})$. This shows the isometric isomorphisms

$$\mathcal{G}_n^{\{\alpha\}} \cong \bigotimes_{0 \leq j \leq n} \mathcal{S}_\alpha^\alpha(\mathbb{R}) \quad \text{and} \quad \mathcal{G}_n^{(\alpha)} \cong \bigotimes_{0 \leq j \leq n} \Sigma_\alpha^\alpha(\mathbb{R}).$$

Moreover, the isometric isomorphisms

$$\mathcal{G}_n^{\{\alpha\}'} \cong \bigotimes_{0 \leq j \leq n} \mathcal{S}_\alpha^\alpha(\mathbb{R})' \quad \text{and} \quad \mathcal{G}_n^{(\alpha)'} \cong \bigotimes_{0 \leq j \leq n} \Sigma_\alpha^\alpha(\mathbb{R})'$$

yield from duality arguments underlying Banach spaces. \square

Let us now make a short parenthesis about the concept of modulation space in time-frequency analysis: Accordingly to [28, Chapter 11], for each $1 \leq p \leq \infty$ the modulation space $M_{\mathbf{m}_{a,\alpha}}^p$ with weight $\mathbf{m}_{a,\alpha}(x, \omega)$ consists on the space of all tempered distributions $f \in \mathcal{S}(\mathbb{R})'$ such that $\|f\|_{M_{\mathbf{m}_{a,\alpha}}^p} := \|\mathbf{m}_{a,\alpha}(\cdot, \cdot) V_\psi f\|_{L^p(\mathbb{R}^2)}$ is finite and independent of the choice of ψ . When $\mathbf{m}_{a,\alpha}(x, \omega) = e^{a(|x|^2 + |\omega|^2)^{\frac{1}{2\alpha}}}$ one can get a *weaker* characterization for $M_{\mathbf{m}_{a,\alpha}}^p$ in terms of $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R})'$ (cf. [29, Proposition 4.1] and [40, Section 4]).

In this way, choosing ψ in the range of (normalized) Hermite functions defined in (14), the characterization of $\mathcal{G}_n^{(\alpha)}$ resp. $\mathcal{G}_n^{\{\alpha\}}$ and its duals as inductive/projective limits involving $\mathcal{W}_{a,\alpha}^{p,n}$ resp. $\mathcal{W}_{-a,\alpha}^{p,n}$ can be obtained by mimecking the result obtained by Teofanov (cf. [40, Theorem 4.3]).

Proposition 4.3. Let $1 \leq p \leq \infty$. Then we have

$$\begin{aligned} \mathcal{G}_n^{(\alpha)} &= \bigcap_{a>0} \mathcal{W}_{a,\alpha}^{p,n}, \quad \mathcal{G}_n^{(\alpha)'} = \bigcup_{a>0} \mathcal{W}_{-a,\alpha}^{p,n} \\ \mathcal{G}_n^{\{\alpha\}} &= \bigcup_{a>0} \mathcal{W}_{a,\alpha}^{p,n}, \quad \mathcal{G}_n^{\{\alpha\}'} = \bigcap_{a>0} \mathcal{W}_{-a,\alpha}^{p,n}. \end{aligned}$$

Proof: Since for any $\sigma \in L^\infty(\mathbb{C})$ such that

$$(P^j \sigma)(x + i\omega, x - i\omega) = e^{-i\pi x \omega} e^{\frac{\pi}{2}(x^2 + \omega^2)} (V_{h_j} f_j)(x, -\omega)$$

(see equation (15)) the quantities $\sum_{j=0}^n \|\mathbf{m}_{a,\alpha}(\cdot, \cdot) V_{h_j} f_j\|_{L^p(\mathbb{R}^2)}$ and $\|\sigma\|_{\mathcal{W}_{a,\alpha}^{p,n}}$ coincide, the proof of Theorem 4.3 follows straightforwardly from Theorem 4.2 and [40, Theorem 4.3]. \square

Remark 4.4. It is clear from the above proposition that these spaces satisfy the following quadruple of imbeddings $\mathcal{G}_n^{(\alpha)} \hookrightarrow \mathcal{G}_n^{\{\alpha\}} \hookrightarrow \mathcal{G}_n^{\{\alpha\}'} \hookrightarrow \mathcal{G}_n^{(\alpha)'}$.

Among this weighted function spaces employed, we will take $\mathcal{W}_{a,\alpha}^{\infty,n}$ for the class of symbols and $\mathcal{W}_{-a,\alpha}^{1,n}$ for the class of windows. From the triplet of imbeddings $\mathcal{W}_{a,\alpha}^{\infty,n} \hookrightarrow \mathbf{F}^n(\mathbb{C}) \hookrightarrow \mathcal{W}_{-a,\alpha}^{1,n}$ we are now able to get a *weaker* formulation of Proposition 2.4. This corresponds to the following result:

Proposition 4.5. For any $\Psi, \Theta \in \mathcal{W}_{-a,\alpha}^{1,n}$ and $\sigma \in \mathcal{W}_{a,\alpha}^{\infty,n}$ there exists $C > 0$ such that $\mathcal{L}_\sigma^{\Psi,\Theta}$ satisfies the boundeness condition:

$$\|\mathcal{L}_\sigma^{\Psi,\Theta}\| \leq C \|\sigma\|_{\mathcal{W}_{a,\alpha}^{\infty,n}} \|\Psi\|_{\mathcal{W}_{-a,\alpha}^{1,n}} \|\Theta\|_{\mathcal{W}_{-a,\alpha}^{1,n}}.$$

Proof: In order to prove the boundeness condition for $\mathcal{L}_\sigma^{\Psi,\Theta}$, recall first the following boundeness result for the *Gabor-Daubechies* operator $\mathcal{A}_\mathbf{a}^{\psi_k, \theta_j}$ obtained by *Cordero & Gröchenig* in [18] in terms of modulation spaces $M_{\mathbf{m}_{a,\alpha}}^\infty$ and $M_{1/\mathbf{m}_{a,\alpha}}^1$ (cf. [18, Theorem 3.2]):

For each $\mathbf{a} \in M_{\mathbf{m}_{a,\alpha}}^\infty$ and $\psi_k, \theta_j \in M_{1/\mathbf{m}_{a,\alpha}}^1$ there exists a constant $C_{j,k} > 0$ such that

$$\|\mathcal{A}_\mathbf{a}^{\psi_k, \theta_j}\| \leq C_{j,k} \|\mathbf{a}\|_{M_{\mathbf{m}_{a,\alpha}}^\infty} \|\psi_k\|_{M_{1/\mathbf{m}_{a,\alpha}}^1} \|\theta_j\|_{M_{1/\mathbf{m}_{a,\alpha}}^1}.$$

Therefore for any $\Psi, \Theta \in \mathcal{W}_{-a,\alpha}^{1,n}$ and $\sigma \in \mathcal{W}_{a,\alpha}^{\infty,n}$ such that $P^k \Psi = \mathcal{B}^k \psi_k$, $P^j \Theta = \mathcal{B}^j \theta_j$ and $\sigma(z, \bar{z}) = \mathbf{a}(\Re(z), \Im(\bar{z}))$, Theorem 2.3 gives the following isometry relation:

$$\mathcal{L}_\sigma^{\Psi,\Theta} = \sum_{j,k=0}^n \mathcal{B}^k \mathcal{A}_\mathbf{a}^{\mathcal{B}^k \psi_k, \mathcal{B}^j \theta_j} (\mathcal{B}^j)^\dagger.$$

Finally, the identities $\|P^k \Psi\|_{\mathcal{W}_{-a,\alpha}^{1,n}} = \|\psi_k\|_{M_{1/\mathbf{m}_{a,\alpha}}^1}$, $\|P^j \Theta\|_{\mathcal{W}_{-a,\alpha}^{1,n}} = \|\theta_j\|_{M_{1/\mathbf{m}_{a,\alpha}}^1}$ triangle's inequality gives

$$\begin{aligned} \|\mathcal{L}_\sigma^{\Psi,\Theta}\| &\leq \sum_{j,k=0}^n C_{j,k} \|\mathbf{a}\|_{M_{\mathbf{m}_{a,\alpha}}^\infty} \|\psi_j\|_{M_{1/\mathbf{m}_{a,\alpha}}^1} \|\psi_k\|_{M_{1/\mathbf{m}_{a,\alpha}}^1} \\ &= \sum_{j,k=0}^n C_{j,k} \|\sigma\|_{\mathcal{W}_{a,\alpha}^{\infty,n}} \|P^k \Psi\|_{\mathcal{W}_{-a,\alpha}^{1,n}} \|P^j \Theta\|_{\mathcal{W}_{-a,\alpha}^{1,n}} \\ &\leq C \|\sigma\|_{\mathcal{W}_{a,\alpha}^{\infty,n}} \|\Psi\|_{\mathcal{W}_{-a,\alpha}^{1,n}} \|\Theta\|_{\mathcal{W}_{-a,\alpha}^{1,n}}, \end{aligned}$$

with $C = \max_{0 \leq j,k \leq n} C_{j,k}$. \square

4.2. Coburn Conjecture Revisited

Now we will extend the framework obtained in Section 3.2 using for the class of windows the space of tempered ultradistributions $\mathcal{G}_n^{(\alpha)'}$ of *Romieu* type and for the class of symbols the Gel'fand-Shilov type space $\mathcal{G}_n^{(\alpha)}$.

First we will start show that Lemma 3.3 can be extended to $\mathcal{G}^{(\alpha)}$. This is indeed a consequence of the following lemma:

Lemma 4.6 (see Appendix A). *For each $\sigma \in \mathcal{W}_{a,\alpha}^{\infty,n}$, $\Psi, \Theta \in \mathcal{G}_n^{\{1/2\}}$ and $0 \leq j, k \leq n$ the operator $D_{j,k}$ defined on Theorem 1.1 satisfy the following convolution formula on \mathbb{C} :*

$$D_{j,k}\sigma * \left(\overline{P^k \Psi} P^j \Theta e^{-\pi|\cdot|^2} \right) = \sigma * D_{j,k} \left(\overline{P^k \Psi} P^j \Theta e^{-\pi|\cdot|^2} \right).$$

Remark 4.7. *Accordingly to [33, Section 2], the condition $\Psi, \Theta \in \mathcal{G}_n^{\{1/2\}}$ is equivalent to the characterization of $\bigotimes_{0 \leq j \leq n} \mathcal{S}_{1/2}^{1/2}(\mathbb{R})$ in terms of the Fourier-Hermite coefficients $\langle \Psi, \Phi_{j,m} \rangle_{d\mu}$ resp. $\langle \Theta, \Phi_{j,m} \rangle_{d\mu}$ of Ψ resp. Θ .*

Indeed each (normalized) Hermite function h_m defined in (14) belongs to $\mathcal{S}_{1/2}^{1/2}(\mathbb{R})$ assures that $\sum_{j=0}^n \Phi_{j,m} = \sum_{j=0}^n \mathcal{B}^j h_m$ belongs to $\mathcal{G}_n^{\{1/2\}} \cong \bigotimes_{0 \leq j \leq n} \mathcal{S}_{1/2}^{1/2}(\mathbb{R})$.

The next theorem corresponds to a *weaker* version of Theorem 1.1 and Corollary 3.7:

Theorem 4.8. *Let $\Psi, \Theta \in \mathcal{G}_n^{\{1/2\}} \cap \mathbb{C}[z, \bar{z}]$ with $\deg(\Psi), \deg(\Theta) < \infty$. Under the assumptions of Corollary 3.7 underlying Ψ and Θ let us assume that for each $a > 0$ the symbol $\sigma(z, \bar{z})$ belongs to $\mathcal{W}_{a,\alpha}^{\infty,n}$. Then for any $F \in \mathcal{W}_{-a,\alpha}^{1,n}$*

$$\begin{aligned} \text{Toep}_{D_{j,k}\sigma} P^k F &= \mathcal{L}_\sigma^{P^k \Psi, P^j \Theta} (P^k F) \\ \sum_{j=0}^n \text{Toep}_{D_j \sigma} F &= \mathcal{L}_\sigma^{\Psi, \Theta} F. \end{aligned}$$

Moreover $D_{j,k}\sigma, D_j \sigma \in \mathcal{G}_n^{(\alpha)}$ and $F \in \mathcal{G}_n^{\{\alpha\}'}$.

Proof: Since Lemma 4.6 fulfils for any $\sigma \in \mathcal{W}_{a,\alpha}^{\infty,n}$, from Cauchy-Schwarz inequality $(D_{j,k}\sigma)P^k F, (D_j \sigma) F \in L^2(\mathbb{C}, d\mu)$ and hence

$$\text{Toep}_{D_{j,k}\sigma}^j P^k F \in \mathcal{F}^j(\mathbb{C}) \quad \text{and} \quad \text{Toep}_{D_j \sigma}^j F \in \mathcal{F}^j(\mathbb{C}).$$

Applying the sequence of ideas used on the proof of Theorem 1.1 we obtain from Lemma 4.5 that we are under the conditions of Proposition 3.4. Then the operators $D_{j,k}$ and D_j determined by Theorem 1.1 and Corollary 3.7, respectively, satisfy $D_{j,k}\sigma, D_j \sigma \in \mathcal{G}_n^{(\alpha)}$ and also the set of equations

$$\text{Toep}_{D_{j,k}\sigma} P^k F = \mathcal{L}_\sigma^{P^k \Psi, P^j \Theta} P^k F \quad \text{and} \quad \sum_{j=0}^n \text{Toep}_{D_j \sigma}^j F = \mathcal{L}_\sigma^{\Psi, \Theta} F.$$

Moreover the constraint $F \in \mathcal{G}_n^{\{\alpha\}'}$ follows straightforwardly from Proposition 4.3. \square

Remark 4.9. For a general $F \in \mathcal{G}_n^{(\alpha)'}$ the functions $(D_{j,k}\sigma)P^k F$ and $(D_j\sigma) F$ do not belong to $L^2(\mathbb{C}, d\mu)$ in general, and thus, $\text{Toep}_{D_{j,k}\sigma}^j P^j F$ likewise $\text{Toep}_{D_j\sigma}^j F$ are not necessary bounded on $\mathcal{F}^j(\mathbb{C})$.

However for $F = \mathbf{K}_z^n = \sum_{j=0}^n K_z^j$, where K_z^j is the normalized reproducing kernel of $\mathcal{F}^j(\mathbb{C})$ obtained in (28), we are under the conditions of Theorem 4.8 since by construction $\sigma \mathbf{K}_z^n \in \mathcal{G}_n^{\{\alpha\}}$ and $\mathcal{G}_n^{\{\alpha\}} \subset L^2(\mathbb{C}, d\mu)$. Thus Theorem 4.8 also fulfils for $F \in \mathcal{G}_n^{(\alpha)'}$ whenever F is a linear combination in terms of $\mathbf{K}_z^n = \sum_{k=0}^n K_z^k$, with $z \in \mathbb{C}$.

In conclusion, this approach is a refinement of Lo's (see [34, Theorem 4.5 & Corollary 4.6]) and Engliš's approach (see [22, Theorem 5]) since the imbedding argument $L^\infty(\mathbb{C}) \hookrightarrow L_{\text{loc}}^2(\mathbb{C}) \hookrightarrow C^\infty(\mathbb{C})$ (cf. [37], Theorem 7.25) assures that the symbol classes $E(\mathbb{C})$ and \mathcal{M}_r belong to $\mathcal{G}_n^{(\alpha)'}$ for any $r \in \mathbb{N}$. Moreover this approach also includes an intriguing characterization for the window classes in terms of Gel'fand-Shilov type spaces of order $\frac{1}{2} \leq \alpha \leq 1$.

5. Acknowledgment

I am grateful to *L.D. Abreu* for calling my attention to this interesting conjecture. I am also grateful to *L.V. Pessoa* and *Z. Mouayn* for the useful discussions at an early stage of this work.

Appendix A. Proof of Technical Lemmata

Appendix A.1. Proof of Lemma 2.3

Proof: Recall that for each $0 \leq j, k \leq n$ the operator \mathcal{B}^j resp. \mathcal{B}^k maps isometrically $L^2(\mathbb{R})$ onto $\mathcal{F}^j(\mathbb{C})$ resp. $\mathcal{F}^k(\mathbb{C})$. Combining this with the change of variable $(x, \omega) \mapsto (x, -\omega)$ we can recast $\mathcal{A}_{\mathbf{a}}^{\psi, \theta}$ in the *weak form* as follows:

$$\begin{aligned} \langle \mathcal{A}_{\mathbf{a}}^{\psi, \theta} f, g \rangle_{L^2(\mathbb{R})} &= \int \int_{\mathbb{R}^2} \mathbf{a}(x, -\omega) \langle f, M_{-\omega} T_x \psi \rangle_{L^2(\mathbb{R})} \langle M_{-\omega} T_x \theta, g \rangle_{L^2(\mathbb{R})} - dx d\omega \\ &= \int \int_{\mathbb{R}^2} \mathbf{a}(x, -\omega) \langle \mathcal{B}^k f, \mathcal{B}^k (M_{-\omega} T_x \psi) \rangle_{d\mu} \langle \mathcal{B}^j (M_{-\omega} T_x \theta), \mathcal{B}^j g \rangle_{d\mu} d\omega dx \\ &= \int \int_{\mathbb{R}^2} \mathbf{a}(x, -\omega) \langle \mathcal{B}^k f, \beta_{x-i\omega}(\mathcal{B}^k \psi) \rangle_{d\mu} \langle \beta_{x-i\omega}(\mathcal{B}^j \theta), \mathcal{B}^j g \rangle_{d\mu} d\omega dx. \end{aligned}$$

Now set $z = x + i\omega$, $\sigma(z, \bar{z}) = \mathbf{a}(x, -\omega)$ and take

$$F = \mathcal{B}^k f, \quad \Psi = \mathcal{B}^k \psi, \quad \Theta = \mathcal{B}^j \theta, \quad G = \mathcal{B}^j g.$$

From the relations $x = \Re(z)$, $-\omega = \Im(\bar{z})$ and $d\omega dx = \frac{dz d\bar{z}}{2i}$ the right-hand side of the above formula can be expressed as the following integration formula over \mathbb{C} with respect to $d^2 z$:

$$\begin{aligned} \langle \mathcal{A}_{\mathbf{a}}^{\psi, \theta} f, g \rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{C}} \sigma(z, \bar{z}) \langle F, \beta_{\bar{z}} \Psi \rangle_{d\mu} \langle \beta_{\bar{z}} \Theta, G \rangle_{d\mu} d^2 z \\ &= \int_{\mathbb{C}} \sigma(z, \bar{z}) e^{-i\pi \Re(\bar{z}) \Im(\bar{z})} \langle F, W_z \Psi \rangle_{d\mu} e^{i\pi \Re(\bar{z}) \Im(\bar{z})} \langle W_z \Theta, G \rangle_{d\mu} d^2 z. \end{aligned}$$

The above equation is equivalent to $\langle \mathcal{A}_{\mathbf{a}}^{\psi, \theta} f, g \rangle_{L^2(\mathbb{R})} = \langle \mathcal{L}_{\sigma}^{\Psi, \Theta} F, G \rangle_{d\mu}$ and therefore $\mathcal{B}^k \mathcal{A}_{\mathbf{a}}^{\psi, \theta} (\mathcal{B}^j)^{\dagger} = \mathcal{L}_{\sigma}^{\mathcal{B}^k \psi, \mathcal{B}^j \phi}$, as desired.

Finally, the proof of relations $\mathcal{L}_{\sigma}^{\mathbf{B}^n \vec{\psi}, \mathbf{B}^j \theta_j} = \sum_{k=0}^n \mathcal{B}^k \mathcal{A}_{\mathbf{a}}^{\psi_k, \theta} (\mathcal{B}^j)^{\dagger}$ and $\mathcal{L}_{\sigma}^{\mathbf{B}^n \psi, \mathbf{B}^n \theta} = \sum_{j,k=0}^n \mathcal{B}^k \mathcal{A}_{\mathbf{a}}^{\psi_k, \theta} (\mathcal{B}^j)^{\dagger}$ follows from combination of definition (21) with linearity arguments. \square

Appendix A.2. Proof of Lemma 3.3

Proof: Starting from definition, straightforward computations combining property (29) with the change of variable $z \mapsto \zeta - z$ results into

$$\begin{aligned} \left(\widetilde{\mathcal{L}_{\sigma}^{\Psi, \Theta}}(\zeta) \right)_{j,k} &= \int_{\mathbb{C}} \sigma(z, \bar{z}) \langle K_{\zeta}^k, W_z \Psi \rangle_{d\mu} \langle W_z \Theta, K_{\zeta}^j \rangle_{d\mu} d^2 z \\ &= \int_{\mathbb{C}} \sigma(z, \bar{z}) \overline{\langle \Psi, W_z^{\dagger} K_{\zeta}^k \rangle_{d\mu}} \langle \Theta, W_z^{\dagger} K_{\zeta}^j \rangle_{d\mu} d^2 z \\ &= \int_{\mathbb{C}} \sigma(z, \bar{z}) \overline{\langle \Psi, K_{\zeta-z}^k \rangle_{d\mu}} \langle \Theta, K_{\zeta-z}^j \rangle_{d\mu} d^2 z \\ &= \int_{\mathbb{C}} \sigma(\zeta - z, \bar{\zeta} - \bar{z}) \overline{\langle \Psi, K^k(\cdot, z) \rangle_{d\mu}} \langle \Theta, K^j(\cdot, z) \rangle_{d\mu} e^{-\pi|z|^2} d^2 z. \end{aligned}$$

Finally, the reproducing kernel property (26) shows that the later integral coincides with $\left[\sigma * \left(\widetilde{\Psi} \Theta e^{-\pi|\cdot|^2} \right) \right](\zeta, \bar{\zeta})$.

Moreover, the proof of relation $\left(\widetilde{\mathcal{L}_{\sigma}^{\Phi_{k,k}, \Phi_{j,j}}}(\zeta) \right)_{j,k} = \left(\widetilde{\text{Toep}_{D_{j,k}\sigma}^j}(\zeta) \right)_{j,k}$ follows straightforwardly from (27). \square

Appendix A.3. Proof of Lemma 3.6

Proof: From the Weyl-Heisenberg relations (8) it follows straightforwardly that $[\pi z, -\frac{1}{4\pi}\Delta_z] = \partial_{\bar{z}}$ and $[\pi \bar{z}, -\frac{1}{4\pi}\Delta_z] = \partial_z$ holds on $L^2(\mathbb{C}, d\mu)$. This leads to

$$\left[\pi z, e^{-\frac{1}{4\pi}\Delta_z} \right] = \partial_{\bar{z}} e^{-\frac{1}{4\pi}\Delta_z} \quad \text{and} \quad \left[\pi \bar{z}, e^{-\frac{1}{4\pi}\Delta_z} \right] = \partial_z e^{-\frac{1}{4\pi}\Delta_z},$$

or equivalently,

$$(\pi \bar{z} - \partial_z) e^{-\frac{1}{4\pi}\Delta_z} = e^{-\frac{1}{4\pi}\Delta_z} (\pi \bar{z}) \quad \text{and} \quad (\pi z - \partial_{\bar{z}}) e^{-\frac{1}{4\pi}\Delta_z} = e^{-\frac{1}{4\pi}\Delta_z} (\pi z).$$

Multiplying both sides of the above identities on the right by the operator $e^{\frac{1}{4\pi}\Delta_z}$, induction over $m \in \mathbb{N}$ completes the proof of Lemma 3.6. \square

Appendix A.4. Proof of Lemma 4.6

Proof: Recall that raising resp. lowering properties (18) resp. (19) shows that $(\partial_z - \pi \bar{z})^r (\partial_{\bar{z}})^l \Phi_{j,k}(z, \bar{z}) = \sqrt{\frac{(j-r)!}{(j-l)!}} \Phi_{j-l+r,k}(z, \bar{z})$ while $(1 + \pi|z|)^m < e^{\pi|z|} \leq e^{\pi|z|^{\frac{1}{\alpha}}}$ shows that $(1 + \pi|z|)^m (P^j \sigma)(z, \bar{z})$ belongs to $\mathcal{W}_{a-\pi, \alpha}^{\infty, n}$.

Therefore $(\partial_z - \pi\bar{z})^r (\partial_{\bar{z}})^l K^j(z, \zeta) = \sqrt{\frac{(j-r)!}{(j-l)!}} K^{j-m+l}(z, \zeta)$, and hence, to the sequence of identities

$$\begin{aligned} (\partial_{\bar{z}})^l (\partial_z)^m (P^j \sigma)(z, \bar{z}) &= \sum_{r=0}^m \binom{m}{r} (\pi\bar{z})^{m-r} (\partial_z - \pi\bar{z})^r (\partial_{\bar{z}})^l (P^j \sigma) \\ &= \sum_{r=0}^m \binom{m}{r} (\pi\bar{z})^{m-r} \int_{\mathbb{C}} \sigma(\zeta, \bar{\zeta}) (\partial_z - \pi\bar{z})^r (\partial_{\bar{z}})^l K^j(z, \zeta) d\mu(\zeta) \\ &= \sum_{r=0}^m \binom{m}{r} (\pi\bar{z})^{m-r} \sqrt{\frac{(j-r)!}{(j-l)!}} (P^{j-m+l} \sigma)(z, \bar{z}). \end{aligned}$$

Thus, the sequence of estimates

$$\begin{aligned} |(\partial_{\bar{z}})^l (\partial_z)^m \sigma(z, \bar{z})| &\leq \sum_{r=0}^m \binom{m}{r} (\pi|z|)^{m-r} \sqrt{\frac{j!}{(j-l)!}} |(P^{j-m+l} \sigma)(z, \bar{z})| \\ &= \sum_{j=m-l}^n \sqrt{\frac{j!}{(j-l)!}} (1 + \pi|z|)^m |(P^{j-m+l} \sigma)(z, \bar{z})|. \end{aligned}$$

yields $(\partial_{\bar{z}})^l (\partial_z)^m \sigma \in \mathcal{W}_{a-\pi, \alpha}^{\infty, n}$ for any $l, m \in \mathbb{N}_0$.

Now let us assume the constraint $\Psi, \Theta \in \mathcal{G}_n^{\{1/2\}}$. Then for any $l, m \in \mathbb{N}_0$ and for some $b, d, C > 0$ we obtain the following upper estimate:

$$\left| (\partial_{\bar{z}})^l (\partial_z)^m \sigma(\zeta - z, \bar{\zeta} - \bar{z}) \overline{P^k \Psi(z, \bar{z})} P^j \Theta(z, \bar{z}) e^{-\pi|z|^2} \right| \leq C e^{\frac{\pi}{2}|\zeta-z|^2 - \frac{\pi}{2}|z|^2} e^{-(a-\pi)|\zeta-z|^{\frac{1}{\alpha}} - (b+d)|z|^2}.$$

The term $e^{\frac{\pi}{2}|\zeta-z|^2 - \frac{\pi}{2}|z|^2} e^{-(a-\pi)|\zeta-z|^{\frac{1}{\alpha}} - (b+d)|z|^2}$ is integrable on \mathbb{C} and satisfies the limit condition $\lim_{|z| \rightarrow \infty} e^{\frac{\pi}{2}|\zeta-z|^2 - \frac{\pi}{2}|z|^2} e^{-(a-\pi)|\zeta-z|^{\frac{1}{\alpha}} - (b+d)|z|^2} = 0$.

This combined with integration by parts gives

$$\begin{aligned} \int_{\mathbb{C}} \partial_{\bar{z}} \left(e^{\frac{\pi}{2}|\zeta-z|^2 - \frac{\pi}{2}|z|^2} e^{-(a-\pi)|\zeta-z|^{\frac{1}{\alpha}} - (b+d)|z|^2} \right) d^2 z &= 0 \\ \int_{\mathbb{C}} \partial_z \left(e^{\frac{\pi}{2}|\zeta-z|^2 - \frac{\pi}{2}|z|^2} e^{-(a-\pi)|\zeta-z|^{\frac{1}{\alpha}} - (b+d)|z|^2} \right) d^2 z &= 0, \end{aligned}$$

and hence, induction over $l, m \in \mathbb{N}_0$ results into the convolution formula

$$(-\partial_{\bar{z}})^l (-\partial_z)^m \sigma * \left(\overline{P^k \Psi} P^j \Theta e^{-\pi|\cdot|^2} \right) = \sigma * \left((-\partial_{\bar{z}})^l (-\partial_z)^m \overline{P^k \Psi} P^j \Theta e^{-\pi|\cdot|^2} \right).$$

Finally, from linearity arguments, the operators $D_{j,k}$ defined on the last section satisfy $D_{j,k} \sigma * \left(\overline{P^k \Psi} P^j \Theta e^{-\pi|\cdot|^2} \right) = \sigma * D_{j,k} \left(\overline{P^k \Psi} P^j \Theta e^{-\pi|\cdot|^2} \right)$. \square

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